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On a Lower Bound for the Absolute Value of a Polynomial of Several Complex Variables

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For an arbitrary polynomial $P(z_1, z_2, \dots, z_n)$ in complex space $\mathbb{C}^n$ we describe a set of nonnegative multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that for any n-tuple $\delta = (\delta_1, \delta_2, \dots, \delta_n) \geq 0$ (where $\delta_j = 0$ if $\alpha_j = 0$), one can find a system of "thin" sets M_j of widths $\leq \delta_j$ in directions of the axes z_j, respectively, $1 \leq j \leq n$, for which outside their union the absolute value of a polynomial is bounded away from zero by $(\delta/\alpha)^\alpha \Gamma_\alpha$ (Γ_α depends on P but not on δ). The prototype of this result is the well known Cartan's Theorem on a lower bound for the modulus of a polynomial $P(z)$, $z \in \mathbb{C}^1$. © 1994 Academic Press, Inc.

1. INTRODUCTION

The localization of roots of analytic functions and polynomials, in particular, plays an important role in various branches of analysis. Results in this area are interesting in and of themselves and can also be applied to the theory of differential equations, to the theory of holomorphic functions, to the general theory of differentiable mappings, etc. This localization problem is obviously related to the problem of describing domains on which the absolute value of a polynomial is bounded away from zero by a positive constant. The sizes and shapes of these domains are very important in applications. Perhaps one of the first results which associates this constant with the distance between the domain and the roots of a polynomial is the following:

CARTAN'S THEOREM [1]. *Let $a_1, a_2, \dots, a_m$ be arbitrary complex numbers. Given any positive δ one can find in the complex plane $\mathbb{C}$ a system M_δ of m circles with radii $\delta_1, \delta_2, \dots, \delta_m$ where $\delta_1 + \delta_2 + \cdots + \delta_m \leq 2\delta$ such that for all points $z \in \mathbb{C} \setminus M_\delta$ the inequality*

$$|z - a_1|\,|z - a_2| \cdots |z - a_m| \geq (\delta/m)^m m! > (\delta/e)^m$$

holds.

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A number of applications of this theorem to the theory of entire functions of one variable can be found, for instance, in [2].

It is obvious that a literal translation of this theorem to the case of a polynomial of several variables $P(z) = P(z_1, z_2, \dots, z_n)$, $z \in \mathbb{C}^n$, is impossible owing to the noncompactness of the set of the roots of $P(z)$ (even in the case of polynomials $P(x_1, x_2, \dots, x_n)$ of real variables). Nevertheless—and it is the main goal of this work—with an arbitrary polynomial $P(z)$, $z \in \mathbb{C}^n$, we associate a set $\mathfrak{A}(P)$ of nonnegative multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that for any n-tuple $\delta = (\delta_1, \delta_2, \dots, \delta_n) \geq 0$, where $\delta_j = 0$ if $\alpha_j = 0$, one can find a system of "thin" mutually disjoint sets M_j of widths $\leq \delta_j$ in the directions of the axes z_j, respectively, $1 \leq j \leq n$, for which outside of $M_{\alpha,\delta} = \bigcup_1^n M_j$ the inequality $|P(z)| > (\delta/\alpha)^\alpha \Gamma_\alpha$ is valid with some positive constants Γ_α.

For obvious reasons one can treat this result as providing information on the location of the polynomial polyhedron

$$\mathfrak{M}_\varepsilon(P) = \{z \in \mathbb{C}^n : |P(z)| < \varepsilon\}.$$

In fact, choosing any multi-index $\alpha \in \mathfrak{A}(P)$ and n-tuple $(\delta_1, \delta_2, \dots, \delta_n)$ and representing the number ε in a form $\varepsilon = (\delta/\alpha)^\alpha \Gamma_\alpha$ we define the corresponding "thin" set $M_{\alpha,\delta}$. It is obvious that $\mathfrak{M}_\varepsilon(P) \subset M_{\alpha,\delta}$.

With respect to applications, we first mention that the results of this work are a crucial step in obtaining *sharp* estimates of the form

$$\|Q(\xi)u(\xi)\|_p \geq K(\Omega)\|u(\xi)\|_p, \qquad u \in L_{p,\Omega}$$

for all function $u(\xi)$ with finite L_p-norm $\|u\|_p$, $1 \leq p \leq \infty$, whose Fourier transforms are compactly supported in a bounded domain $\Omega \subset \mathbb{R}^n$. The weight function $Q(\xi)$ belongs to a wide class which includes for example all functions $q(|P(\xi)|)$ with arbitrary polynomial P and arbitrary nondecreasing q. What is most important is that the constant $K(\Omega)$ is explicitly expressed in terms of geometric characteristics of Ω. This is done in [3].

On the other hand, as it follows from some facts noted in [3], there is undoubtedly a connection of the results of this paper with multidimensional variants of the well known Remez's Theorem (cf. [4]). We hope to return to this connection at a later date.

2. Results

We denote by $z = (z_1, z_2, \dots, z_n)$ an arbitrary point of the n-dimensional complex space $\mathbb{C}^n$ and for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of length $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, with nonnegative integers α_j, we put

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \quad \text{and} \quad \partial^\alpha = (\partial/\partial z_1)^{\alpha_1}(\partial/\partial z_2)^{\alpha_2} \cdots (\partial/\partial z_n)^{\alpha_n}.$$

Let $P(z) = \sum_{|\alpha| \le m} a_\alpha z^\alpha$ be an arbitrary polynomial with complex coefficients a_α. Denote By $\mathcal{N}_P$ the set of all multi-indices α for which $a_\alpha \neq 0$.

DEFINITION. Given a polynomial $P(z) = \sum a_\alpha z^\alpha$ we call a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ a *leading* multi-index of $P(z)$ if

(i) $\partial^\alpha P(z) \equiv \text{const} \neq 0$,

(ii) $(\partial/\partial z_1)^{\alpha_1}(\partial/\partial z_2)^{\alpha_2} \cdots (\partial/\partial z_j)^{\alpha_j+1} P(z) \equiv 0$

for all $j = 1, 2, \dots, n$ such that $\alpha_j \neq 0$.

The set of all leading indices of the polynomial $P(z)$ we denote by $\mathfrak{A}(P)$. Note that according to (i) the set $\mathfrak{A}(P)$ is always nonempty. To see this, assume that the first $j_1 - 1$ components $\alpha_1, \alpha_2, \dots, \alpha_{j_1-1}$ of all multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{N}_P$ are equal to zero and $\alpha_{j_1} \neq 0$ for some $\alpha \in \mathcal{N}_P$. Let $\alpha_{j_1}^0 = \max_{\alpha \in \mathcal{N}_P} \alpha_{j_1}$. Define the set $\mathcal{N}_1 \subset \mathcal{N}_P$ of multi-indices $\alpha = (0, \dots, 0, \alpha_{j_1}^0, \dots)$. Assume that for every $\alpha \in \mathcal{N}_1$ we have $\alpha_j = 0$, $j_1 < j < j_2$, and $\alpha_{j_2} \neq 0$ for at least one of these α. Let $\alpha_{j_2}^0 = \max_{\alpha \in \mathcal{N}_1} \alpha_{j_2}$ and define the set $\mathcal{N}_2 \subset \mathcal{N}_1$ of multi-indices $\alpha = (0, \dots, 0, \alpha_{j_1}^0, 0, \dots, 0, \alpha_{j_2}^0, \dots)$. Continuing this procedure we find a multi-index $\alpha = (0, \dots, 0, \alpha_{j_1}^0, 0, \dots, 0, \alpha_{j_2}^0, 0, \dots, 0, \alpha_{j_k}^0, \dots)$ which obviously belongs to $\mathfrak{A}(P)$.

EXAMPLE. For the polynomial $P(z) = z_1^3 + z_2^3 + z_3^2 + z_1^2 z_2^2 z_3$ among all multi-indices (3, 0, 0), (0, 3, 0), (0, 0, 2), (2, 2, 1) (only) the last multi-index is not a leading multi-index of $P(z)$, although the corresponding monomial $z_1^2 z_2^2 z_3$ is the principal part of the polynomial $P(z)$.

In the space $\mathbb{C}^n$ of variables $z = (z_1, z_2, \dots, z_n)$ put $\hat{z}_j = (z_1, z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ for all $j = 1, 2, \dots, n$. For every $\hat{a}_j \in \mathbb{C}^{n-1}$ denote by $\mathscr{C}(\hat{a}_j)$ the complex straight line $\{z \in \mathbb{C}^n\colon \hat{z}_j = \hat{a}_j,\ z_j \in \mathbb{C}\}$. Further, for arbitrary nonnegative n-tuples $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with integers α_j, and $\delta = (\delta_1, \delta_2, \dots, \delta_n)$ put

$$(\delta/\alpha)^\alpha = (\delta_1/\alpha_1)^{\alpha_1}(\delta_2/\alpha_2)^{\alpha_2} \cdots (\delta_n/\alpha_n)^{\alpha_n} \quad \text{and}$$
$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!.$$

THEOREM 1. *Let $P(z)$ be an arbitrary polynomial of degree m in the space $\mathbb{C}^n$ and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be any leading index of $P(z)$, i.e., $\alpha \in \mathfrak{A}(P)$. Given any nonnegative n-tuple $\delta = (\delta_1, \delta_2, \dots, \delta_n)$ with $\delta_j = 0$ iff $\alpha_j = 0$, $1 \le j \le n$, there exists a system of sets $M_1, M_2, \dots, M_n$ in $\mathbb{C}^n$ such that all of them are mutually disjoint, each of the M_j intersects with any* *complex straight line* $\mathscr{C}(\hat{a}_j)$ *on* $m_j \le \alpha_j$ *circles with the sum of radii* $\le 2\delta_j$, *and*

$$|P(z)| \ge (\delta/\alpha)^{\alpha} |\partial^{\alpha} P|$$

for all points $z \in \mathbb{C}^n \setminus M_{\alpha,\delta}$, *where* $M_{\alpha,\delta} = \bigcup_{j=1}^{n} M_j$.

It is obvious that for $n = 1$ this is exactly Cartan's Theorem.

Proof. The proof is based on a representation of a polynomial $P(z_1, z_2, \dots, z_n)$ in a special form, which corresponds to a fixed leading index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. This representation allows us to construct the needed set $M_{\alpha,\delta}$ by means of k-multiple application of Cartan's Theorem (where k is equal to the number of nonvanishing components of α). We may, without loss of generality, assume that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_q, 0, \dots, 0)$, $q \le n$, where all $\alpha_j > 0$, $1 \le j \le q$. In fact, if all components of the multi-index α are equal to zero except for $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_q}$ we introduce in the space $\mathbb{C}^n$ new coordinates $(\zeta_1, \zeta_2, \dots, \zeta_n)$ such that $\zeta_j = z_{\alpha_j}$ if $j = 1, 2, \dots, q$ and $\zeta_k = z_{\beta_k}$ for other $\beta_k \ne \alpha_j$ if $k = q+1, q+2, \dots, n$. Set

$$z'_j = (z_j, z_{j+1}, \dots, z_q) \text{ if } j \le q; \qquad z'' = (z_{q+1}, z_{q+2}, \dots, z_n),$$

where each variable z_k, $1 \le k \le n$, runs over the complex plane $\mathbb{C}$. The space of the points z'' we denote by $\mathbb{C}_{n-q}$. For every *fixed* $z'_{j+1} \in \mathbb{C}^{q-j}$ put

$$Z_j(z'_{j+1}) = \left\{(z_j, z'_{j+1}) \in \mathbb{C}^{q-j+1}, z_j \in \mathbb{C}\right\}$$

if $1 \le j \le q-1$. The set $Z_j(z'_{j+1})$ is a complex straight line in the space $\mathbb{C}^{q-j+1}$ (which can be considered as a line parallel to the z_j axes in $\mathbb{C}^{q-j+1}$). Let $\hat{\mathscr{K}}_j(z'_{j+1})$ be an arbitrary system of $m_j \le \alpha_j$ circles in $Z_j(z'_{j+1})$ and let $\mathscr{K}_j(z'_{j+1}) = Z_j(z'_{j+1}) \setminus \hat{\mathscr{K}}_j(z'_{j+1})$. Let Z_q be the plane of the variable z_q and $\hat{\mathscr{K}}_q$, $\mathscr{K}_q$ are defined analogously to $\hat{\mathscr{K}}_j$ and $\mathscr{K}_j$, $j < q$. If $1 \le j \le q-1$ then for an arbitrary nonempty set $\mathscr{C} \subset \mathbb{C}^{q-j}$ we denote by $\hat{\mathscr{K}}_j \circ \mathscr{C}$ (respectively $\mathscr{K}_j \circ \mathscr{C}$) a subset in the complex space of the variables $z'_j = (z_j, z_{j+1}, \dots, z_q)$ consisting of all points $(z_j, z'_{j+1}) \in \hat{\mathscr{K}}_j(z'_{j+1})$ (respectively $(z_j, z'_{j+1}) \in \mathscr{K}_j(z'_{j+1})$) with $z'_{j+1} \in \mathscr{C}$. In other words,

$$\hat{\mathscr{K}}_j \circ \mathscr{C} = \bigcup_{z'_{j+1} \in \mathscr{C}} \hat{\mathscr{K}}_j(z'_{j+1}), \qquad 1 \le j \le q-1,$$

and analogously for $\mathscr{K}_j \circ \mathscr{C}$. Define inductively the sets

$$\hat{G}_q = \hat{\mathscr{K}}_q, \qquad \hat{G}_{q-1} = \hat{\mathscr{K}}_{q-1} \circ \mathscr{K}_q, \qquad \hat{G}_{q-2} = \hat{\mathscr{K}}_{q-1} \circ (\mathscr{K}_{q-1} \circ \mathscr{K}_{q-2}), \dots$$
$$\hat{G}_1 = \hat{\mathscr{K}}_1 \circ (\mathscr{K}_2 \circ \cdots \circ (\mathscr{K}_q) \cdots)$$

and

$$G_q = \mathscr{K}_q, \qquad G_{q-1} = \mathscr{K}_{q-1} \circ \mathscr{K}_q, \dots, G_1 = \mathscr{K}_1 \circ \left(\mathscr{K}_2 \circ \cdots \circ (\mathscr{K}_q) \cdots \right)$$

in the spaces of variables

$$z'_q = z_q, \qquad z'_{q-1} = (z_{q-1}, z_q), \qquad z'_{q-2} = (z_{q-2}, z_{q-1}, z_q), \dots, z'_1 = z,$$

respectively. Let $\mathbb{C}^{j-1}$ denote the space of the complementary variables $(z_1, z_2, \dots, z_{j-1})$, $2 \le j \le q-1$, and $\mathbb{C}^0 = \phi$. It is easy to verify that

$$G_1 = \mathbb{C}^q \setminus \bigcup_{j=1}^{q} \left(\mathbb{C}^{j-1} \times \hat{G}_j\right).$$

Put

$$M_\alpha = G_1 \times \mathbb{C}_{n-q}.$$

Then

$$M_\alpha = \mathbb{C}^n \setminus \bigcup_{j=1}^{q} M_j,$$

where all the sets $M_j = \mathbb{C}^{j-1} \times \hat{G}_j \times \mathbb{C}_{n-q}$ are mutually disjoint and each of them is intersected by any complex straight line $\mathscr{C}(\hat{a}_j)$ along a union of $m_j \le \alpha_j$ circles, $1 \le j \le q$.

Now let us return to our polynomial $P(z) = P(z_1, z_2, \dots, z_n)$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_q, 0, \dots, 0)$ be an arbitrary leading index and $\delta = (\delta_1, \delta_2, \dots, \delta_q)$ be a positive q-tuple, $q \le n$. Then from the definition of the leading multi-index it follows that there exists a unique system of polynomials $P_2(z'_2, z'')$, $P_3(z'_3, z''), \dots, P_q(z_q, z'')$ such that

$$P(z) =: P_1(z_1, z'_2, z'') = A z_1^{\alpha_1} P_2(z'_2, z'') + \sum_{\beta > 0} z_1^{\alpha_1 - \beta} P_{1\beta}(z'_2, z'')$$

$$P_2(z'_2, z'') = z_2^{\alpha_2} P_3(z'_3, z'') + \sum_{\beta > 0} z_2^{\alpha_2 - \beta} P_{2\beta}(z'_3, z'')$$

$$P_{q-1}(z'_{q-1}, z'') = z_{q-1}^{\alpha_{q-1}} P_q(z_q, z'') + \sum_{\beta > 0} z_{q-1}^{\alpha_{q-1} - \beta} P_{q-1,\beta}(z_q, z'')$$

$$P_q(z_q, z'') = z_q^{\alpha_q} + \sum_{\beta > 0} z_q^{\alpha_k - \beta} P_{q\beta}(z''),$$

where $A = \partial^\alpha P/\alpha!$. Using this representation choose at first a set $\hat{\mathscr{K}}_q$ such that for all points $z_q \in \mathscr{K}_q = G_q$ the inequality

$$|P_q(z_q, z'')| > (\delta_q/\alpha_q)^{\alpha_q} \alpha_q!$$

is valid. The possibility of such a choice is guaranteed by Cartan's Theorem. Further, for an arbitrary point $z_q \in G_q$ we can choose (using Cartan's Theorem again) a set $\mathscr{K}_{q-1}(z_q)$ such that at any point $z'_{q-1} = (z_{q-1}, z_q)$ of the set $G_{q-1} = \mathscr{K}_{q-1} \circ \mathscr{K}_q$ the inequality

$$\left| \frac{P_{q-1}(z_{q-1}, z_q, z'')}{P_q(z_q, z'')} \right| > (\delta_{q-1}/\alpha_{q-1})^{\alpha_{q-1}} \alpha_{q-1}!$$

holds. Continuing this process we successively obtain for $l = q-2, q-3, \ldots, 1$, a system of sets G_l such that at any point $z'_l \in G_l$ the inequality

$$\left| \frac{P_l(z_l, z'_{l+1}, z'')}{P_{l+1}(z'_{l+1}, z'')} \right| > (\delta_l/\alpha_l)^{\alpha_l} \alpha_l!$$

is valid for all $z'' \in \mathbb{C}_{n-q}$. Combining these inequalities we find that for all points $z'_l \in G_l$

$$|P_l(z'_l, z'')| \geq \prod_{k=l}^{q} (\delta_k/\alpha_k)^{\alpha_k} \alpha_k!.$$

But this means that for all points $z'_1 = (z_1, z_2, \ldots, z_q) \in G_1$ and arbitrary $z'' \in \mathbb{C}_{n-q}$ we have the inequality

$$|P(z)| \geq |A|(\delta/\alpha)^{\alpha} \alpha! = |\partial^{\alpha} P|(\delta/\alpha)^{\alpha}.$$

Consider the set

$$M_{\alpha,\delta} = G_1 \times \mathbb{C}_{n-q}.$$

By the definition of G_1 this set satisfies all needed geometric conditions of the theorem and at each point $z \in M_{\alpha,\delta}$ the inequality of the theorem holds. This completes the proof.

Remark 1. The result of Theorem 1 is also true for complex valued polynomials $P(x) = P(x_1, x_2, \ldots, x_n)$ defined on the real space $\mathbb{R}^n$. The sets M_j, $1 \leq j \leq n$, in this case are just units of some strip-shaped domains in $\mathbb{R}^n$ and each of the M_j intersects with any x_j-directed straight line on a one-dimensional set of Lebesgue measure $\leq 4\delta_j$. The proof in this case is given in [3].

Remark 2. Theorem 1 also gives a certain description of the polynomial polyhedron

$$\mathfrak{M}_\delta(P) = \{z \in \mathbb{C}^n : |P(z)| < \delta\}.$$

It is worth noting that unlike with polynomials on $\mathbb{R}^n$, the set $\mathfrak{M}_\delta(P)$ contains information on the location of roots of the polynomial $P(z)$ in the space $\mathbb{C}^n$. In fact, the well known polynomial $P(x_1, x_2) = (x_1 x_2 - 1)^2 + x_1^2$ has no roots in $\mathbb{R}^n$, but the equation $P(x_1, x_2) = \delta$ is solvable for an arbitrary small $\delta > 0$. So the inequality $|P(\mathring{x}_1, \mathring{x}_2)| < \delta$ with $\delta \ll 1$ tells us nothing about how close the point $(\mathring{x}_1, \mathring{x}_2)$ is to the set $\mathcal{N}_P = \{(x_1, x_2) \in \mathbb{R}^n : P(x_1, x_2) = 0\}$. But *for an arbitrary polynomial* $P(z)$ *on* $\mathbb{C}^n$ *the inequality* $|P(\mathring{z})| < \delta$ *allows us to conclude that* $\operatorname{dist}(\mathring{z}, \mathcal{N}_P) < c\delta^{1/m}$, *where* m *is the degree of* $P(z)$, $\mathcal{N}_P = \{z \in \mathbb{C}^n : P(z) = 0\}$ *and* c *is a constant which does not depend on* $\mathring{z}$ *and* δ. In fact, after some unitary transformation $\eta: z \to \zeta$ in $\mathbb{C}^n$ we shall obtain the polynomial $Q(\zeta) = P(\eta^* \zeta)$ which can be written as

$$Q(\zeta) = c^m \zeta_1^m + \zeta_1^{m-1} Q_{m-1}(\zeta') + \cdots + Q_0(\zeta'),$$

where $\zeta' = (\zeta_2, \zeta_3, \ldots, \zeta_n)$, all Q_j are polynomials and c is a constant. Let

$$Q = c^m(\zeta_1 - \varphi_1(\zeta'))(\zeta_1 - \varphi_2(\zeta')) \cdots (\zeta_1 - \varphi_m(\zeta'))$$

be the factorization of Q for some ζ'. Put $\mathring{\zeta} = \eta \mathring{z}$. Then $|Q(\mathring{\zeta})| < \delta$ and so at least one of the factors $|\mathring{\zeta}_1 - \varphi_j(\mathring{\zeta}')|$ is majorized by $c^{-1}\delta^{1/m}$. But the point $\hat{\zeta} = (\varphi_j(\mathring{\zeta}'), \mathring{\zeta}')$ belongs to the set $\mathcal{N}_Q$ of the roots of the polynomial Q and $|\mathring{\zeta} - \hat{\zeta}| = |\mathring{\zeta}_1 - \varphi_j(\mathring{\zeta}')| < c^{-1}\delta^{1/m}$. Since the mapping η is unitary it follows that for $\hat{z} = \eta^* \hat{\zeta}$

$$\operatorname{dist}(\mathring{z}, \mathcal{N}_P) \leq |\mathring{z} - \hat{z}| < c^{-1}\delta^{1/m}.$$

Since the set $\mathfrak{M}_\delta(P)$ arises in various applications very often we reformulate (in a more general form) Theorem 1 in connection with $\mathfrak{M}_\delta(P)$. Let us introduce Stiefel's manifold $\mathfrak{N}$ of all orthonormal bases $\eta = \{\eta_1, \eta_2, \ldots, \eta_n\}$ in the space $\mathbb{C}^n$. Put

$$\partial_{\eta_j} P(z) = \langle \eta_j, \operatorname{grad}_z P(z) \rangle, \qquad 1 \leq j \leq n,$$

where $\langle\ ,\ \rangle$ is the scalar product in Euclidean space $\mathbb{C}^n$. For an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ put $\partial_\eta^\alpha = \partial_{\eta_1}^{\alpha_1} \partial_{\eta_2}^{\alpha_2} \cdots \partial_{\eta_n}^{\alpha_n}$ (if $\alpha_j = 0$ then $\partial_{\eta_j}^{\alpha_j}$ is omitted). As before, we call a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *the* *leading multi-index of $P(z)$ with respect to a basis η* if

$$\partial_\eta^\alpha P(z) \equiv \text{const} \neq 0 \qquad \text{and} \qquad \partial_{\eta_1}^{\alpha_1} \partial_{\eta_2}^{\alpha_2} \cdots \partial_{\eta_j}^{\alpha_j + 1} P(z) \equiv 0$$

for all j such that $\alpha_j \neq 0$.

THEOREM 1′. *Let $P(z)$, $\delta > 0$, and $\mathfrak{M}_\delta(P)$ be the same as previously. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a leading index of $P(z)$ with respect to a basis $\eta = \{\eta_1, \eta_2, \ldots, \eta_n\} \in \mathfrak{N}$. Choose an arbitrary nonnegative n-tuple $\hat{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ in which $\delta_j = 0$ iff $\alpha_j = 0$ and such that $\delta = (\hat{\delta}/\alpha)^\alpha |\partial_\eta^\alpha P|$. Then the set $\mathfrak{M}_\delta(P)$ is imbedded in the union $\bigcup M_j$ where each of the sets M_j intersects with every complex straight line $\mathscr{C}_j$ which is parallel to the vector η_j on a system of $m_j \leq \alpha_j$ circles with the sum of radii $\leq 2\delta_j$.*

The details of the proof may be easily reconstructed by the reader.

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